Case IV - Conicoids of Revolutions -

When λ - cubic has represented non-zero root

Sub case I : Two roots of λ - cubic are equal and third root non-zero.

Form -

$A(x^2+y^2)+Bz^2=1$ (ellipsoid of Revolution)

$A(x^2-y^2)+Bz^2=1$ (Hyperboloid of Revolution)

$A(x^2+y^2)+Bz^2=0$ (cone of Revolution)

Sub case 2 : Two roots of λ - cubic are equal and third root is zero.

Form -

$A(x^2+y^2)+Bz^2=0$ (paraboloid of Revolution)

$A(x^2+y^2)+D=0$ (Right circular cylinder)

Sub case 3 : Three roots of λ - cubic are equal and each is non-zero.

Form -

The equation represents sphere

Ex-1 P.T $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$ represents a paraboloid of revolution. Find the coordinates of its focus and equation to its axes.

→ The given equation is

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0 \quad \text{--- (1)}$$

comparing it with

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$ we get

$a=1$ $\quad f=\frac{-1}{2}$ $\quad u=\frac{-3}{2}$ $\quad d=21$

$b=1$ $\quad g=\frac{-1}{2}$ $\quad v=-3$

$c=1$ $\quad h=\frac{-1}{2}$ $\quad w=\frac{-9}{2}$

$$A=bc-f^2 = 1-\frac{1}{4}=\frac{3}{4}$$

$$C=ab-h^2 = 1-\frac{1}{4}=\frac{3}{4}$$

$$B=ca-g^2 = 1-\frac{1}{4}=\frac{3}{4}$$

$$D=abc+2fgh-af^2-bg^2-ch^2$$
$$=1-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}$$
$$=1-\frac{4}{4}=1-1=0$$

The discriminating cubic is

$$\lambda^3 - (a+b+c)\lambda^2 + (A+B+C)\lambda - \Delta = 0$$

$$\lambda^3 - (1+1+1)\lambda^2 + \left(\frac{3}{4}+\frac{3}{4}+\frac{3}{4}\right)\lambda - 0 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\left(\frac{3}{4}\right)\lambda = 0$$

$$\lambda\left(\lambda^2 - 3\lambda + \frac{9}{4}\right) = 0$$

$$\lambda\left(\lambda - \frac{3}{2}\right)^2 = 0$$

$$\lambda = \frac{3}{2}, \frac{3}{2}, 0$$

$\therefore$ two values of λ are equal

$\therefore$ the quadratic is Surface of a Revolution

The direction cosines l, m, n of the principal direction corresponding to $\lambda = 0$ are given by two of the equations.

$$\Rightarrow \quad l - \frac{1}{2}m - \frac{1}{2}n = 0$$

$$2l - m - n = 0$$

$$\Rightarrow \quad -\frac{l}{2} + m - \frac{n}{2} = 0$$

$$-l + 2m - n = 0$$

$$\Rightarrow \quad -\frac{l}{2} - \frac{m}{2} + n = 0$$

$$-l - m + 2n = 0$$

Solving first two equations, we get

$$\frac{l}{1+2} = \frac{m}{1+2} = \frac{n}{4-1}$$

$$\frac{l}{3} = \frac{m}{3} = \frac{n}{3}$$

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}}$$

$$= \frac{1}{\sqrt{3}}$$

$$l = \frac{1}{\sqrt{3}}, \quad m = \frac{1}{\sqrt{3}}, \quad n = \frac{1}{\sqrt{3}}$$

Also, $ul + vm + wn$

$$= -\frac{3}{2} \cdot \frac{1}{\sqrt{3}} - 3 \cdot \frac{1}{\sqrt{3}} - \frac{9}{2} \cdot \frac{1}{\sqrt{3}}$$

$$= \frac{-3}{2\sqrt{3}} - \frac{3}{\sqrt{3}} - \frac{9}{2\sqrt{3}}$$

$$= \frac{-\cancel{12}\,6}{\cancel{2}\sqrt{3}} - \frac{3}{\sqrt{3}}$$

$$= \frac{-6}{\sqrt{3}} - \frac{3}{\sqrt{3}}$$

$$= \frac{-9}{\sqrt{3}} \neq 0$$

$\therefore$ the Surface is a paraboloid of Revolution and Reduced equation is

$$\lambda_1 x^2 + \lambda_1 y^2 + 2Kz = 0$$

$$\frac{3}{2} x^2 + \frac{3}{2} y^2 - 2 \cdot \frac{9}{\sqrt{3}} z = 0$$

$$x^2 + y^2 = 4\sqrt{3}\, z \quad \text{———} \quad (5)$$

The vertex is given by equations

$$\frac{\frac{\partial f}{\partial x}}{2l} = \frac{\frac{\partial f}{\partial y}}{2m} = \frac{\frac{\partial f}{\partial z}}{2n} = k$$

$$x - \frac{1}{2}y - \frac{1}{2}z - \frac{3}{2} + 3\sqrt{3}\left(\frac{1}{\sqrt{3}}\right) = 0$$

$$-\frac{1}{2}x + y - \frac{1}{2}z - 3 + \sqrt{3} + \left(\frac{1}{\sqrt{3}}\right) = 0$$

and $$-3\sqrt{3}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right) - \frac{3}{2}x - 3y - \frac{9}{2}z + 21 =$$

$$2x - y - z + 3 = 0$$

$$-x + 2y - z = 0$$

$$3x + 4y + 5z - 14 = 0$$

Putting $x = 0$ in first two equations

$$-y - z + 3 = 0$$

$$2y - z = 0$$

Subtracting we get $3y - 3 = 0$

$$y = 1$$

$2 - z = 0$

$z = 2$

we have $x = 0, \quad y = 1, \quad z = 2$

vertex is $(0, 1, 2)$

$\therefore$ equations to the axes are.

$$\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1}$$

$$x = y - 1 = z - 2$$

from (2) the latus Rectum of the generating parabola is $4\sqrt{3}$.

The focus will be a point on axes at distance $\frac{1}{4}(4\sqrt{3})$ or $\sqrt{3}$ from vertex $(0, 1, 2)$

The coordinates are given by

$$\frac{x}{\frac{1}{\sqrt{3}}} = \frac{y-1}{\frac{1}{\sqrt{3}}} = \frac{z-2}{\frac{1}{\sqrt{3}}} = \sqrt{3}$$

coordinates of focus are $(1, 2, 3)$